Algebraic Methods in Dynamical Systems

Nonlinear and Complex Systems Group Research Programme

In the realm of Dynamical Systems, an important problem is to decide whether a given system is "solvable" in a reasonably simple manner. If so, the system is usually called *integrable*. Otherwise, its evolution with respect to time is generally unpredictable and very sensible to initial conditions – a phenomenon commonly known as *chaos*. One concept is not intrinsically antonymous with the other, but there seems to be indeed an inverse correlation between the two in practical examples.

If the system is *Hamiltonian*, as are most problems in Mechanics, this "chaos vs solvability" disjunctive is doubly advantageous. On one hand, the techniques of symplectic geometry may be adapted to our situation. On the other hand, in virtue of the empirical studies and the Liouville-Arnold theory, the notion of "integrability" has been rendered equivalent to a very specific, and thus observable, condition: the existence of a customary amount of independent first integrals in pairwise involution.

The algebraic approaches by Ziglin, Morales-Ruiz and Ramis are a major breakthrough in the study of Hamiltonian integrability. It is based on the study of the invariants of a given matrix group – be it the monodromy or the differential Galois group. Each of these invariants arises from one of the first integrals of the original dynamical system. Said matrix group is linked to the *first-order variational equations* VE_Γ along a given integral curve $\Gamma = \{\widehat{\boldsymbol{z}}(t) : t \in I \subset \mathbb{C}\}$. A second step forward was done by Morales-Ruiz, Ramis and Simó in order to extend the preceeding theoretical framework to the Galois groups of the *(linearized) higher-order variational systems* $\left\{\mathrm{LVE}_\Gamma^k : k \geq 1\right\}$.

Using this theory on the first-order VE_Γ, we have so far proven the non-integrability of a certain number of problems in Celestial Mechanics: Hill's Problem, the Three-Body Problem (already proven non-integrable by other means by Tsygvintsev, Boucher and Weil) and the equal-mass N-Body Problem, as well as established necessary conditions on the existence of additional first integrals for Hamiltonians of the form $H = \boldsymbol{p}^T\boldsymbol{p}/2 + V(\boldsymbol{q})$ with a homogeneous potential V. Since the N-Body Problem fits this profile, these conditions implied the absence of an additional first integral for the Three-Body Problem and the equal-mass $4, 5, 6$- Body Problems.

We currently study the higher-order Galois groups $G_k = \mathrm{Gal}\left\{\mathrm{LVE}_\Gamma^k\right\}$ and the link between their inverse limit $\widehat{G} = \varprojlim G_k$ and the *Galois groupoid* of the system as defined by Malgrange and Umemura. Stokes and monodromy matrices are studied separately within every G_k. This is being done for homogeneous potentials so far, the more general case being the next logical step.

Keywords: Dynamical systems, Hamiltonian systems, chaos theory, integrability, homogeneous potentials, invariant theory of algebraic groups, Conley index theory, spectral sequences, differential Galois theory, Morales-Ramis-Ziglin theory, groupoids and categories, Godbillon-Vey invariants, first integrals, tensor invariants, Celestial Mechanics, N-Body Problem, central configurations.

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